Planck's Law and the Light Quantum Hypothesis

From Satyendra Nath **Bose** (Dacca-University, India)

(Received on 2 July 1924.)

The phase space of a light quantum with respect to a given volume is divided into "cells" of size h^3. The number of possible distributions of the light quanta of a macroscopically defined radiation among these cells provides the entropy and thus all the thermodynamic properties of the radiation.

Planck's formula for the distribution of energy in black body radiation forms the starting point for quantum theory, which has been developed over the last 20 years and has borne rich fruit in all areas of physics. Since its publication in 1901, many ways of deriving this law have been proposed. It is recognized that the fundamental assumptions of quantum theory are incompatible with the laws of classical electrodynamics. All previous derivations make use of the relation

$$\rho_\nu d\nu = \frac{8\pi\nu^2 d\nu}{c^3} E,$$

that is, about the relation between the radiation density and the average energy of an oscillator. They also make assumptions about the number of degrees of freedom of the aether, as included in the above equation (first factor on the right-hand side). However, this factor could only be derived from classical theory. This is the unsatisfactory point in all derivations, and it is not surprising that efforts are continually made to give a derivation which is free from this logical error.

A remarkably elegant derivation was given by Einstein. He recognized the logical deficiency of all previous derivations and tried to deduce the formula independently of the classical theory. Starting from very simple assumptions about the energy exchange between molecules and the radiation field, he finds the relation

$$\rho_\nu = \frac{\alpha_{mn}}{e^{\frac{\epsilon_m-\epsilon_n}{kT}} - 1}.$$

However, in order to bring this formula into agreement with Planck's, he must make use of Wien's displacement law and Bohr's correspondence principle. Wien's law is based on classical theory, and the correspondence principle assumes that quantum theory agrees with classical theory in certain borderline cases.

In all cases, the derivations do not seem to me to be sufficiently logically justified. On the other hand, the light quantum hypothesis in conjunction with statistical mechanics (as Planck adapted it to the needs of quantum theory) seems to me to be sufficient for the derivation of the law independently of the classical theory. In the following I would like to briefly outline the method.

The radiation is enclosed in the volume V and its total energy E is given. There are different types of quanta with the respective number N_S, and energy $h\nu_S$ ($S = 0$ to $S = \infty$). The total energy E is then

$$E = \sum_s N_s h\nu_s = V \int \rho_\nu d\nu \,. \tag{1}$$

The solution to the problem then requires the determination of the N_S, which determines ρ_ν. If we can specify the probability for any distribution characterized by arbitrary N_S, then the solution is determined by the condition that this probability should be a maximum while maintaining the secondary condition (1). We now want to look for this probability.

The quantum has a moment of magnitude $\frac{h\nu_S}{c}$ in the direction of its movement. The instantaneous state of the quantum is characterized by its coordinates x, y, z and the associated moments p_x, p_y, p_z. These six quantities can be thought of as point coordinates in a six-dimensional space, where we have the relationship

$$p_x^2 + p_y^2 + p_z^2 = \frac{h^2\nu^2}{c^2} \,,$$

by virtue of which the point mentioned is forced to remain on a cylindrical surface determined by the frequency of the quantum. In this sense, the phase space belongs to the frequency range $d\nu_S$

$$\int dx\, dy\, dz\, dp_x\, dp_y\, dp_z = V \cdot 4\pi \left(\frac{h\nu}{c}\right)^2 \frac{hd_\nu}{c} = 4\pi \frac{h^3\nu^2}{c^3} V d\nu$$

If we divide the entire phase volume into cells of size h^3, the $d\nu$ frequency range therefore includes $4\pi V \frac{\nu^2}{c^3}$ cells. Nothing definite can be said as to the nature of this division. Meanwhile, the total number of cells must be viewed as the number of possible arrangements of a quantum in the given volume. In order to take the fact of polarization into account, it seems advisable to multiply this number by 2, so that we get $8\pi V\frac{\nu^2 d\nu}{c^3}$ for the number of cells belonging to $d\nu$.

Now it is easy to calculate the thermodynamic probability of a (macroscopically defined) state. Let N^S be the number of quanta belonging to the frequency range $d\nu^S$. In how many ways can these be distributed among the cells belonging to $d\nu^S$? Let p_0^S be the number of vacant cells, p_1^S the number of those containing one quantum, p_2^S the number of cells containing two quanta, etc. The number of possible distributions is then

$$\frac{A^S!}{p_0^S!\,p_1^S!\ldots}\,, \qquad \text{where} \qquad A^S = \frac{8\pi\nu^2}{c^3} V\,d\nu^S\,,$$

and where

$$N^S = 0\cdot p_0^S + 1\cdot p_1^S + 2p_2^S \ldots$$

is the number of quanta belonging to $d\nu^S$.

The probability of the state defined by all p_r^S is obviously

$$\prod_S \frac{A^S!}{p_0^S!\,p_1^S!\ldots}\,.$$

Considering that we can consider the p_r^S as large numbers, we have

$$\ln W = \sum_S A^S lnA^S - \sum_S\sum_r p_r^S \ln p_r^S\,,$$

where

$$A^S = \sum_r p_r^S\,.$$

This expression should be a maximum under the constraint

$$E = \sum_S N^S h\nu^S\,; \qquad N^S = \sum_r r p_r^S\,.$$

Carrying out the variation provides the conditions

$$\sum_S\sum_r \delta p_r^S(1+\ln p_r^S) = 0\,, \qquad \sum_S \delta N^S h\nu^S = 0$$

$$\sum_r \delta p_r^S = 0 \qquad \delta N^S = \sum_r r\delta p_r^S\,.$$

From this it follows that

$$\sum_S\sum_r \delta p_r^S(1+\ln p_r^S+\lambda^S) + \frac{1}{\beta}\sum_S h\nu^S\sum_r r\delta p_r^S = 0\,.$$

From this it first follows that

$$p_r^S = B^S e^{-\frac{rh\nu^S}{\beta}}.$$

But since

$$A^S = \sum_r B^S e^{-\frac{rh\nu^S}{\beta}} = B^S \left(1 - e^{-\frac{h\nu^S}{\beta}}\right)^{-1},$$

then

$$B_S = A^S \left(1 - e^{-\frac{h\nu^S}{\beta}}\right).$$

Furthermore, we have

$$N^S = \sum_r r p_r^S = \sum_r A^S \left(1 - e^{-\frac{h\nu^S}{\beta}}\right) e^{-\frac{rh\nu^S}{\beta}}$$

$$= \frac{A^S e^{-\frac{h\nu^S}{\beta}}}{1 - e^{-\frac{h\nu^S}{\beta}}}.$$

So, taking into account the value of A^S found above,

$$E = \sum_S \frac{8\pi h{\nu^S}^3 d\nu^S}{c^3} V \frac{e^{-\frac{h\nu^S}{\beta}}}{1 - e^{-\frac{h\nu^S}{\beta}}}.$$

Using the previous results one also finds

$$S = k\left[\frac{E}{\beta} - \sum_S A^S \ln\left(1 - e^{\frac{h\nu^S}{\beta}}\right)\right],$$

from which, considering that $\beta \frac{\partial S}{\partial E} = \frac{1}{T}$, it follows that $\beta = kT$. Inserting this result into the above equation for E gives

$$E = \sum_S \frac{8\pi h{\nu^S}^3}{c^3} V \frac{1}{e^{\frac{h\nu^S}{kT}} - 1} = d\nu^S,$$

which is equivalent to Planck's formula.

(Translated by A. Einstein)

Comment by translator. Bose's derivation of Planck's formula appears to me to be an important step forward. The method used here gives also the quantum theory of an ideal gas, as I shall show elsewhere.